

# Linear Receding Horizon Control with Probabilistic System Parameters

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**Abstract:** In this paper we address the problem of designing receding horizon control algorithms for linear discrete-time systems with parametric uncertainty. We do not consider presence of stochastic forcing or process noise in the system. It is assumed that the parametric uncertainty is probabilistic in nature with known probability density functions. We use generalized polynomial chaos theory to design the proposed stochastic receding horizon control algorithms. In this framework, the stochastic problem is converted to a deterministic problem in higher dimensional space. The performance of the proposed receding horizon control algorithms is assessed using a linear model with two states.

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## 1. INTRODUCTION

Receding horizon control (RHC), also known as model predictive control (MPC), has been popular in the process control industry for several years Qin and Badgwell [1996], Bemporad and Morari [1999], and recently gaining popularity in aerospace applications, see Bhattacharya et al. [2002]. It is based on the idea of repetitive solution of an optimal control problem and updating states with the first input of the optimal command sequence. The repetitive nature of the algorithm results in a state dependent feedback control law. The attractive aspect of this method is the ability to incorporate state and control limits as constraints in the optimization formulation. When the model is linear, the optimization problem is quadratic if the performance index is expressed via a  $\mathcal{L}_2$ -norm, or linear if expressed via a  $\mathcal{L}_1/\mathcal{L}_\infty$ -norm. Issues regarding feasibility of online computation, stability and performance are largely understood for linear systems and can be found in refs. Kwon [1994], Bitmead et al. [1990]. For nonlinear systems, stability of RHC methods is guaranteed by Primbs [1999], Jadbabaie et al. [1999], by using an appropriate control Lyapunov function. For a survey of the state-of-the-art in nonlinear receding horizon control problems the reader is directed to Mayne et al. [2000a].

Traditional RHC laws perform best when modeling error is small. Fisher et al. [2007] has shown that system uncertainty can lead to significant oscillatory behavior and possibly instability. Furthermore, Grimm et al. [2004] showed that in the presence of modeling uncertainty RHC strategy may not be robust with RHC designs. Many approaches have been taken to improve robustness of RHC strategy in the presence of unknown disturbances and bounded uncertainty, see work of Raković et al. [2006], Lee and Yu [1997], Kouvaritakis et al. [2000], Mayne et al. [2000b]. These approaches involve the computation of a feedback gain to ensure robustness. The difficulty with this approach is that, even for linear systems, the problem becomes difficult to solve, as the unknown feedback gain transforms the quadratic programming problem into a nonlinear programming problem.

In this paper we address the problem of RHC design for linear systems with probabilistic uncertainty in system parameters. Parametric uncertainty arises in systems when the physics governing the system is known and the system parameters are either not known precisely or are expected to vary in the operational lifetime. Such uncertainty also occurs when system models are built from experimental data using system identification techniques. As a result of experimental measurements, the values of the parameters in the system model have a range of variations with quantifiable likelihood of occurrence. In either case, the range of variation of these parameters and the likelihood of their occurrence are assumed to be known and it is desired to design controllers that achieve specified performance for these variations.

While the area of robust RHC is not new, approaching the problem from a stochastic standpoint is only recently receiving attention, for example van Hessem and Bosgra [2002], Batina et al. [2002]. These approaches however suffered from either computational complexity, high degree of conservativeness or do not address closed-loop stability. The key difficulty in stochastic RHC is the propagation of uncertainty over the prediction horizon. More recently, Cannon et al. [2009] avoid this difficulty by using an autonomous augmented formulation of the prediction dynamics. Constraint satisfaction and stability is achieved in Cannon et al. [2009] by extending ellipsoid invariance theory to invariance with a given probability. The cost function minimized was the expected value of a quadratic function of random state and control trajectories. Additionally, the uncertainty in the system parameters were assumed to have normal distribution.

This paper presents formulation of robust RHC design problems in polynomial chaos framework, where parametric uncertainty can be governed by any probability density function. In this approach the solution, not the dynamics, of the random process is approximated using a series expansion. It is assumed that the random process to be controlled has finite second moment, which is the assumption of the polynomial chaos framework. The polynomial

chaos based approach predicts the propagation of uncertainty more accurately, is computationally cheaper than methods based on Monte-Carlo or series approximation of the dynamics, and is less conservative than the invariance based methods.

The paper is organized as follows. We first present a brief introduction to polynomial chaos and its application in transforming linear stochastic dynamics to linear deterministic dynamics in higher dimensional state-space. Next stability of stochastic linear dynamics in the polynomial chaos framework is presented. This is followed by formulation of RHC design for discrete-time stochastic linear systems. Stability of the proposed RHC algorithm is then analyzed. The paper concludes with numerical examples that assesses the performance of the proposed method.

## 2. BACKGROUND ON POLYNOMIAL CHAOS

Recently, use of polynomial chaos to study stochastic differential equations is gaining popularity. It is a non-sampling based method to determine evolution of uncertainty in dynamical system, when there is probabilistic uncertainty in the system parameters. Polynomial chaos was first introduced by Wiener [1938] where Hermite polynomials were used to model stochastic processes with Gaussian random variables. It can be thought of as an extension of Volterra's theory of nonlinear functionals Schetzen [2006] for stochastic systems Ghanem and Spanos [1991]. According to Cameron and Martin [1947] such an expansion converges in the  $\mathcal{L}_2$  sense for any arbitrary stochastic process with finite second moment. This applies to most physical systems. Xiu and Karniadakis [2002] generalized the result of Cameron-Martin to various continuous and discrete distributions using orthogonal polynomials from the so called Askey-scheme Askey and Wilson [1985] and demonstrated  $\mathcal{L}_2$  convergence in the corresponding Hilbert functional space. This is popularly known as the generalized polynomial chaos (gPC) framework. The gPC framework has been applied to applications including stochastic fluid dynamics Hou et al. [2006], stochastic finite elements Ghanem and Spanos [1991], and solid mechanics Ghanem and Red-Horse [1999]. It has been shown in Xiu and Karniadakis [2002] that gPC based methods are computationally far superior than Monte-Carlo based methods. However, application of gPC to control related problems has been surprisingly limited and is only recently gaining popularity. See Prabhakar et al. [2008], Fisher and Bhattacharya [2008a,b] for control related application of gPC theory.

### 2.1 Wiener-Askey Polynomial Chaos

Let  $(\Omega, \mathcal{F}, P)$  be a probability space, where  $\Omega$  is the sample space,  $\mathcal{F}$  is the  $\sigma$ -algebra of the subsets of  $\Omega$ , and  $P$  is the probability measure. Let  $\Delta(\omega) = (\Delta_1(\omega), \dots, \Delta_d(\omega)) : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}^d, \mathcal{B}^d)$  be an  $\mathbb{R}^d$ -valued continuous random variable, where  $d \in \mathbb{N}$ , and  $\mathcal{B}^d$  is the  $\sigma$ -algebra of Borel subsets of  $\mathbb{R}^d$ . A general second order process  $X(\omega) \in \mathcal{L}_2(\Omega, \mathcal{F}, P)$  can be expressed by polynomial chaos as

$$X(\omega) = \sum_{i=0}^{\infty} x_i \phi_i(\Delta(\omega)), \quad (1)$$

where  $\omega$  is the random event and  $\phi_i(\Delta(\omega))$  denotes the gPC basis of degree  $p$  in terms of the random variables

$\Delta(\omega)$ . The functions  $\{\phi_i\}$  are a family of orthogonal basis in  $\mathcal{L}_2(\Omega, \mathcal{F}, P)$  satisfying the relation

$$\langle \phi_i \phi_j \rangle := \int_{\mathcal{D}_{\Delta(\omega)}} \phi_i \phi_j w(\Delta(\omega)) d\Delta(\omega) = h_i^2 \delta_{ij} \quad (2)$$

where  $\delta_{ij}$  is the Kronecker delta,  $h_i$  is a constant term corresponding to  $\int_{\mathcal{D}_{\Delta}} \phi_i^2 w(\Delta) d\Delta$ ,  $\mathcal{D}_{\Delta}$  is the domain of the random variable  $\Delta(\omega)$ , and  $w(\Delta)$  is a weighting function. Henceforth, we will use  $\Delta$  to represent  $\Delta(\omega)$ . For random variables  $\Delta$  with certain distributions, the family of orthogonal basis functions  $\{\phi_i\}$  can be chosen in such a way that its weight function has the same form as the probability density function  $f(\Delta)$ . When these types of polynomials are chosen, we have  $f(\Delta) = w(\Delta)$  and

$$\int_{\mathcal{D}_{\Delta}} \phi_i \phi_j f(\Delta) d\Delta = \mathbf{E}[\phi_i \phi_j] = \mathbf{E}[\phi_i^2] \delta_{ij}, \quad (3)$$

where  $\mathbf{E}[\cdot]$  denotes the expectation with respect to the probability measure  $dP(\Delta(\omega)) = f(\Delta(\omega))d\Delta(\omega)$  and probability density function  $f(\Delta(\omega))$ . The orthogonal polynomials that are chosen are the members of the Askey-scheme of polynomials (Askey and Wilson [1985]), which forms a complete basis in the Hilbert space determined by their corresponding support. Table 1 summarizes the correspondence between the choice of polynomials for a given distribution of  $\Delta$ . See Xiu and Karniadakis [2002] for more details.

Random Variable $\Delta$	$\phi_i(\Delta)$ of the Wiener-Askey Scheme
Gaussian	Hermite
Uniform	Legendre
Gamma	Laguerre
Beta	Jacobi

Table 1. Correspondence between choice of polynomials and given distribution of  $\Delta(\omega)$  Xiu and Karniadakis [2002].

### 2.2 Approximation of Stochastic Linear Dynamics Using Polynomial Chaos Expansions

Here we derive a generalized representation of the deterministic dynamics obtained from the stochastic system by approximating the solution with polynomial chaos expansions.

Define a linear discrete-time stochastic system in the following manner

$$x(k+1, \Delta) = A(\Delta)x(k, \Delta) + B(\Delta)u(k, \Delta), \quad (4)$$

where  $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ . The system has probabilistic uncertainty in the system parameters, characterized by  $A(\Delta), B(\Delta)$ , which are matrix functions of random variable  $\Delta \equiv \Delta(\omega) \in \mathbb{R}^d$  with certain *stationary* distributions. Due to the stochastic nature of  $(A, B)$ , the system trajectory  $x(k, \Delta)$  will also be stochastic.

By applying the Wiener-Askey gPC expansion of finite order to  $x(k, \Delta), A(\Delta)$  and  $B(\Delta)$ , we get the following approximations,

$$\hat{x}(k, \Delta) = \sum_{i=0}^p x_i(k) \phi_i(\Delta), \quad x_i(k) \in \mathbb{R}^n \quad (5)$$

$$\hat{u}(k, \Delta) = \sum_{i=0}^p u_i(k) \phi_i(\Delta), \quad u_i(k) \in \mathbb{R}^m \quad (6)$$

$$\hat{A}(\Delta) = \sum_{i=0}^p A_i \phi_i(\Delta), \quad A_i = \frac{\langle A(\Delta), \phi_i(\Delta) \rangle}{\langle \phi_i(\Delta)^2 \rangle} \in \mathbb{R}^{n \times n} \quad (7)$$

$$\hat{B}(\Delta) = \sum_{i=0}^p B_i \phi_i(\Delta), \quad B_i = \frac{\langle B(\Delta), \phi_i(\Delta) \rangle}{\langle \phi_i(\Delta)^2 \rangle} \in \mathbb{R}^{n \times m} \quad (8)$$

The inner product or ensemble average  $\langle \cdot, \cdot \rangle$ , used in the above equations and in the rest of the paper, utilizes the weighting function associated with the assumed probability distribution, as listed in table 1.

The number of terms  $p$  is determined by the dimension  $d$  of  $\Delta$  and the order  $r$  of the orthogonal polynomials  $\{\phi_k\}$ , satisfying  $p+1 = \frac{(d+r)!}{d!r!}$ . The  $n(p+1)$  time varying coefficients,  $\{x_i(k)\}; k=0, \dots, p$ , are obtained by substituting the approximated solution in the governing equation (eqn.(4)) and conducting Galerkin projection on the basis functions  $\{\phi_k\}_{k=0}^p$ , to yield  $n(p+1)$  *deterministic* linear system of equations, which given by

$$\mathbf{X}(k+1) = \mathbf{A}\mathbf{X}(k) + \mathbf{B}\mathbf{U}(k), \quad (9)$$

where

$$\mathbf{X}(k) = [x_0(k)^T \ x_1(k)^T \ \dots \ x_p(k)^T]^T, \quad (10)$$

$$\mathbf{U}(k) = [u_0(k)^T \ u_1(k)^T \ \dots \ u_p(k)^T]^T. \quad (11)$$

$$(12)$$

Matrices  $\mathbf{A} \in \mathbb{R}^{n(p+1) \times n(p+1)}$  and  $\mathbf{B} \in \mathbb{R}^{n(p+1) \times m}$  are defined as

$$\mathbf{A} = (W \otimes I_n)^{-1} \begin{bmatrix} H_A(E_0 \otimes I_n) \\ \vdots \\ H_A(E_p \otimes I_n) \end{bmatrix}, \quad (13)$$

$$\mathbf{B} = (W \otimes I_n)^{-1} \begin{bmatrix} H_B(E_0 \otimes I_m) \\ \vdots \\ H_B(E_p \otimes I_m) \end{bmatrix}, \quad (14)$$

where  $H_A = [A_0 \dots A_p]$ ,  $H_B = [B_0 \dots B_p]$ ,  $W = \text{diag}(\langle \phi_0^2 \rangle, \dots, \langle \phi_p^2 \rangle)$ , and

$$E_i = \begin{bmatrix} \langle \phi_i, \phi_0, \phi_0 \rangle & \dots & \langle \phi_i, \phi_0, \phi_p \rangle \\ \vdots & & \vdots \\ \langle \phi_i, \phi_p, \phi_0 \rangle & \dots & \langle \phi_i, \phi_p, \phi_p \rangle \end{bmatrix},$$

with  $I_n$  and  $I_m$  as the identity matrix of dimension  $n \times n$  and  $m \times m$  respectively. It can be easily shown that  $\mathbf{E}[x(k)] = x_0(k)$ , or  $\mathbf{E}[x(k)] = [I_n \ 0_{n \times np}] \mathbf{X}(k)$ .

Therefore, transformation of a stochastic linear system with  $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ , with  $p^{\text{th}}$  order gPC expansion, results in a *deterministic* linear system with increased dimensionality equal to  $n(p+1)$ .

### 3. STOCHASTIC RECEDING HORIZON CONTROL

Here we develop a RHC methodology for stochastic linear systems similar to that developed for deterministic systems, presented by Goodwin et al. [2005]. Let  $x(k, \Delta)$  be

the solution of the system in eqn.(4) with control  $u(k, \Delta)$ . Consider the following optimal control problem defined by,

$$V_N^* = \min V_N(\{x(k+1, \Delta)\}, \{u(k, \Delta)\}) \quad (15)$$

subject to :

$$x(k+1, \Delta) = A(\Delta)x(k, \Delta) + B(\Delta)u(k, \Delta), \quad (16)$$

$$\text{Initial Condition: } x(0, \Delta); \quad (17)$$

$$\mu(u(k, \Delta)) \in \mathbb{U} \subset \mathbb{R}^m, \quad (18)$$

$$\mu(x(k, \Delta)) \in \mathbb{X} \subset \mathbb{R}^n, \quad (19)$$

$$\mu(x(N, \Delta)) \in \mathbb{X}_f \subset \mathbb{X}, \quad (20)$$

for  $k = 0, \dots, N-1$ ; where  $N$  is the horizon length,  $\mathbb{U}$  and  $\mathbb{X}$  are feasible sets for  $u(k, \Delta)$  and  $x(k, \Delta)$  with respect to control and state constraints.  $\mu(\cdot)$  represents moments based constraints on state and control. The set  $\mathbb{X}_f$  is a terminal constraint set. The cost function  $V_N$  is given by

$$V_N = \sum_{k=1}^N \mathbf{E} [x^T(k, \Delta) Q x(k, \Delta) + u^T(k-1, \Delta) R u(k-1, \Delta)] + C_f(x(N), \Delta), \quad (21)$$

where  $C_f(x(N), \Delta)$  is a terminal cost function, and  $Q = Q^T > 0$ ,  $R = R^T > 0$  are matrices with appropriate dimensions.

#### 3.1 Control Structure

Here we consider the control structure,

$$u(k, \Delta) = \bar{u}(k) + K(k) (x(k, \Delta) - \mathbf{E}[x(k, \Delta)]), \quad (22)$$

where  $\bar{u}(k)$ , and  $K(k)$  are unknown *deterministic* quantities. This is similar to that proposed by Primbs et al. Primbs and Sung [2009] and enables us to regulate the mean trajectory using open loop control and deviations about the mean using a state-feedback control.

In terms of gPC coefficients, the system dynamics in eqn.(4) with the first control structure is given by eqn.(9). The system dynamics in term of the gPC expansions, with the second control structure is given by

$$\mathbf{X}(k+1) = (\mathbf{A} + \mathbf{B}(\mathbf{M} \otimes K(k)))\mathbf{X}(k) + \mathbf{B}\bar{U}(k), \quad (23)$$

where  $\bar{U}(k) = [1 \ 0_{1 \times p}]^T \otimes \bar{u}(k)$  and  $\mathbf{M} = \begin{bmatrix} 0 & 0_{1 \times p} \\ 0_{p \times 1} & I_{p \times p} \end{bmatrix}$ .

#### 3.2 Cost Functions

Here we derive the cost function in eqn.(21) is derived in terms of the gPC coefficients  $\mathbf{X}$  and  $\mathbf{U}$ . For scalar  $x$ , the quantity  $\mathbf{E}[x^2]$  in terms of its gPC expansions is given by

$$\mathbf{E}[x^2] = \sum_{i=0}^p \sum_{j=0}^p x_i x_j \int_{\mathcal{D}_\Delta} \phi_i \phi_j f d\Delta = \mathbf{x}^T W \mathbf{x}, \quad (24)$$

where  $\mathcal{D}_\Delta$  is the domain of  $\Delta$ ,  $x_i$  are the gPC expansions of  $x$ ,  $f \equiv f(\Delta)$  is the probability distribution of  $\Delta$ . Here we use the notation  $\mathbf{x}$  to represent the gPC state vector for scalar  $x$ . The expression  $\mathbf{E}[x^2]$  can be generalized for  $x \in \mathbb{R}^n$  where  $\mathbf{E}[x^T Q x]$  is given by

$$\mathbf{E}[x^T Q x] = \mathbf{X}^T (W \otimes Q) \mathbf{X}. \quad (25)$$

The expression for the cost function in eqn.(21), in terms of gPC states and control is

$$V_N = \sum_{k=0}^{N-1} [\mathbf{X}^T(k) \bar{Q} \mathbf{X}(k) + (\bar{U}^T(k) + \mathbf{X}^T(k)(\mathbf{M} \otimes K^T(k))) \bar{R}(\bar{U}(k) + (\mathbf{M} \otimes K(k)) \mathbf{X}(k))] + C_f(x(N), \Delta), \quad (26)$$

where  $\bar{Q} = W \otimes Q$  and  $\bar{R} = W \otimes R$ .

In deterministic RHC, the terminal cost is the cost-to-go from the terminal state to the origin by the local controller Goodwin et al. [2005]. In the stochastic setting, a local controller can be synthesized using methods presented in our previous work Fisher and Bhattacharya [2008a]. The cost-to-go from a given stochastic state variable  $x(N, \Delta)$  can then be written as

$$C_f(x(N), \Delta) = \mathbf{X}^T(N) P \mathbf{X}(N), \quad (27)$$

where  $\mathbf{X}(N)$  are gPC states corresponding to  $x(N, \Delta)$  and  $P = P^T > 0$  is a  $n(p+1) \times n(p+1)$ -dimensional matrix, obtained from the synthesis of the terminal control law Fisher and Bhattacharya [2008a]. In the current stochastic RHC literature, the terminal cost function has been defined on the expected value of the final state Lee and Cooley [1998], de la Penad et al. [2005], Primbs and Sung [2009], Bertsekas [2005] or using a combination of mean and variance Darlington et al. [2000], Nagy and Braatz [2003]. The terminal cost function in eqn.(27) is more general than the terminal cost functions used in the literature because it penalizes all the moments of the random variable  $x(N, \Delta)$ , as they are functions of  $\mathbf{X}(N)$ . This can be shown as follows.

To avoid tensor notation and without loss of generality, we consider  $x(k, \Delta) \in \mathbb{R}$  and let  $\mathbf{X}(k) = [x_0(k), x_1(k), \dots, x_p(k)]^T$  be the gPC expansion of  $x(k, \Delta)$ . The  $p^{th}$  moment in terms of  $x_i(k)$  are then given by

$$m_p(k) = \sum_{i_1=0}^P \dots \sum_{i_p=0}^P x_{i_1}(k) \dots x_{i_p}(k) \int_{\mathcal{D}_\Delta} \phi_{i_1}(\Delta) \dots \phi_{i_p}(\Delta) f(\Delta) d\Delta. \quad (28)$$

Thus, minimizing  $C_f(x(N), \Delta)$  in eqn.(27) minimizes all moments of  $x(N, \Delta)$ . Consequently, constraining the probability density function of  $x(N, \Delta)$ .

### 3.3 State and Control Constraints

In this section we present the state and control constraints for the receding horizon policy.

*Expectation Based Constraints* Here we first consider constraints of the following form,

$$\mathbf{E}[x(k, \Delta)^T \bar{H}_x x(k, \Delta) + \bar{G}_x x(k, \Delta)] \leq \alpha_{i,x}, \quad (29)$$

$$\mathbf{E}[u(k, \Delta)^T \bar{H}_u u(k, \Delta) + \bar{G}_u u(k, \Delta)] \leq \alpha_{i,u}, \quad (30)$$

for  $k = 0 \dots N$ . These constraints are on the *expected value* of the quadratic functions. Thus, instead of requiring that the constraint be met for all trajectories, they instead imply that the constraints should be satisfied on average. These constraints can be expressed in terms of the gPC states as

$$\mathbf{X}(k)^T \bar{H}_x \mathbf{X}(k) + \bar{G}_x \mathbf{X}(k) \leq \alpha_{i,x}, \quad (31)$$

$$\mathbf{U}(k)^T \bar{H}_u \mathbf{U}(k) + \bar{G}_u \mathbf{U}(k) \leq \alpha_{i,u}, \quad (32)$$

where  $\bar{H}_x = W \otimes H_x$ ,  $\bar{H}_u = W \otimes H_u$ ,  $\bar{G}_x = G_x [I_n \ 0_{n \times np}]$ , and  $\bar{G}_u = G_u [I_n \ 0_{n \times np}]$ .

*Variance Based Constraints* In many practical applications, it may be desirable to constrain the second moment of the state trajectories, either at each time step or at final time. One means of achieving this is to use a constraint of the form

$$\text{Tr}[\mathbf{E}[(x(k) - \mathbf{E}[x(k)])(x(k) - \mathbf{E}[x(k)])^T]] \leq \alpha_{\sigma^2}. \quad (33)$$

For scalar  $x$ , the variance  $\sigma^2(x)$  in terms of the gPC expansions can be shown to be

$$\sigma^2 = \mathbf{E}[x - \mathbf{E}[x]]^2 = \mathbf{E}[x^2] - \mathbf{E}[x]^2 = \mathbf{x}^T W \mathbf{x} - \mathbf{E}[x]^2,$$

where

$$\mathbf{E}[x] = \mathbf{E}\left[\sum_{i=0}^p x_i \phi_i\right] = \sum_{i=0}^p x_i \mathbf{E}[\phi_i] = \sum_{i=0}^p x_i \int_{\mathcal{D}_\Delta} \phi_i f d\Delta = \mathbf{x}^T F,$$

and  $F = [1 \ 0 \ \dots \ 0]^T$ . Therefore,  $\sigma^2$  for scalar  $x$  can be written in a compact form as

$$\sigma^2 = \mathbf{x}^T (W - FF^T) \mathbf{x}. \quad (34)$$

In order to represent the covariance for  $x \in \mathbb{R}^n$ , in terms of the gPC states, let us define  $\Phi = [\phi_0 \dots \phi_{p+1}]^T$  and write  $G = \int_{\mathcal{D}_\Delta} \Phi \Phi^T f d\Delta$ . Let us represent a sub-vector of  $\mathbf{X}$ , defined by elements  $n_1$  to  $n_2$ , as  $X_{n_1 \dots n_2}$ , where  $n_1$  and  $n_2$  are positive integers. Let us next define matrix  $M_{\mathbf{X}}$ , with subvectors of  $\mathbf{X}$ , as  $M_{\mathbf{X}} = [X_{1 \dots n} \ X_{n+1 \dots 2n} \ \dots \ X_{np+1 \dots n(p+1)}]$ . For  $x \in \mathbb{R}^n$ , it can be shown that

$$\mathbf{E}[x] = (F \otimes I_n) \mathbf{X}, \quad (35)$$

and the covariance can then be shown to be

$$\mathbf{Cov}(x) = M_{\mathbf{X}} G M_{\mathbf{X}}^T - (F \otimes I_n) \mathbf{X} \mathbf{X}^T (F^T \otimes I_n). \quad (36)$$

The trace of the covariance matrix  $\mathbf{Cov}(x)$  can then be written as

$$\text{Tr}[\mathbf{Cov}(x)] = \mathbf{X}^T ((W - FF^T) \otimes I_n) \mathbf{X}.$$

Therefore, a constraint of the type

$$\text{Tr}[\mathbf{Cov}(x(k))] \leq \alpha_{\sigma^2}$$

can be written in term of gPC states as

$$\mathbf{X}^T Q_{\sigma^2} \mathbf{X} \leq \alpha_{\sigma^2}, \quad (37)$$

where  $Q_{\sigma^2} = (W - FF^T) \otimes I_n$ .

## 4. STABILITY OF THE RHC POLICY

Here we show the stability properties of the receding horizon policy when it is applied to the system in eqn.(9). Using gPC theory we can convert the underlying stochastic RHC formulation in  $x(t, \Delta)$  and  $u(t, \Delta)$  to a deterministic RHC formulation in  $\mathbf{X}(k)$  and  $\mathbf{U}(k)$ . The stability of  $\mathbf{X}(k)$  in an RHC setting, with suitable terminal controller, can be proved using results by Goodwin et al. [2005], which shows that  $\lim_{k \rightarrow \infty} \mathbf{X}(k) \rightarrow 0$ , when a receding horizon policy is employed. To relate this result to the stability of  $x(k, \Delta)$ , we first present the following known result in stochastic stability in terms of the moments of  $x(k, \Delta)$ . For stochastic dynamical systems in general, stability of moments is a weaker definition of stability than the *almost sure stability* definition. However, the two definitions are equivalent for linear autonomous systems

(pg. 296, Khas'minskii [1969] also pg. 349 Chen and Hsu [1995]). Here we present the definition of asymptotic stability in the  $p^{th}$  moment for discrete-time systems.

*Definition 1.* The zero equilibrium state is said to be stable in the  $p^{th}$  moment if  $\forall \epsilon > 0, \exists \delta > 0$  such that

$$\sup_{k \geq 0} \mathbf{E}[x(k, \Delta)^p] \leq \epsilon, \quad \forall x(0, \Delta) : \|x(0, \Delta)\| \leq \delta, \forall \Delta \in \mathcal{D}_\Delta. \quad (38)$$

*Definition 2.* The zero equilibrium state is said to be asymptotically stable in the  $p^{th}$  moment if it is stable in  $p^{th}$  moment and

$$\lim_{k \rightarrow \infty} \mathbf{E}[x(k, \Delta)^p] = 0, \quad (39)$$

for all  $x(0, \Delta)$  in the neighbourhood of the zero equilibrium.

*Proposition 1.* For the system in eqn.(4),  $\lim_{k \rightarrow \infty} \mathbf{X}(k) \rightarrow 0$  is a sufficient condition for the asymptotic stability of the zero equilibrium state, in all moments.

**Proof** To avoid tensor notation and without loss of generality, we consider  $x(k, \Delta) \in \mathbb{R}$  and let  $\mathbf{X}(k) = [x_0(k), x_1(k), \dots, x_p(k)]^T$  be the gPC expansion of  $x(k, \Delta)$ . The moments in terms of  $x_i(k)$  are given by eqn.(28). Therefore, if  $\lim_{k \rightarrow \infty} \mathbf{X}(k) \rightarrow 0 \implies \lim_{k \rightarrow \infty} x_i(k) \rightarrow 0$ . Consequently,  $\lim_{k \rightarrow \infty} m_i(k) \rightarrow 0$  for  $i = 1, 2, \dots$ , and eqn.(39) is satisfied. This completes the proof.  $\square$

## 5. NUMERICAL EXAMPLE

Here we consider the following linear system, similar to that considered in Primbs and Sung [2009],

$$x(k+1) = (A + G(\Delta))x(k) + Bu(k) \quad (40)$$

where

$$A = \begin{bmatrix} 1.02 & -0.1 \\ .1 & .98 \end{bmatrix}, B = \begin{bmatrix} 0.1 \\ 0.05 \end{bmatrix}, G = \begin{bmatrix} 0.04 & 0 \\ 0 & 0.04 \end{bmatrix} \Delta.$$

The system in consideration is open-loop unstable and the uncertainty appears linearly in the  $G$  matrix. Here,  $\Delta \in [-1, 1]$  and is governed by a uniform distribution, that doesn't change with time. Consequently, Legendre polynomials is used for gPC approximation and polynomials up to 4<sup>th</sup> order are used to formulate the control. Additionally, we assume that there is no uncertainty in the initial condition. The expectation based constraint is imposed on  $x(k, \Delta)$  as

$$\mathbf{E} \left[ \begin{bmatrix} 1 & 0 \end{bmatrix} x(k, \Delta) \right] \geq -1,$$

which in terms of the gPC states, this corresponds to

$$\begin{bmatrix} 1 & \mathbf{0}_{1 \times 2p+1} \end{bmatrix} \mathbf{X}(k) \geq -1.$$

The terminal controller is designed using probabilistic LQR design techniques described by Fisher and Bhattacharya [2008a]. The cost matrices used to determine the terminal controller are

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}, R = 1.$$

Figure (1) illustrates the performance of the proposed RHC policy. The resulting optimization problem is a nonlinear programming problem which has been solved using MATLAB's `fmincon(...)` function. From the figure, we see that the constraint on the expected value of  $x_1$  has been satisfied and the RHC algorithm was able to stabilize the system. These plots have been obtained using 4<sup>th</sup> order gPC approximation of the stochastic dynamics.

## 6. SUMMARY

In this paper we present a RHC strategy for linear discrete time systems with probabilistic system parameters. We have used the polynomial chaos framework to design stochastic RHC algorithms in an equivalent deterministic setting. The controller structure has an open loop component that controls the mean behavior of the system, and a state-feedback component that controls deviations about the mean trajectory. This controller structure results in a polynomial optimization problem with polynomial constraints that is solved in the general nonlinear programming framework. Theoretical guarantees for the stability of the proposed algorithm has also been presented. Performance of the RHC algorithm has been assessed using a two dimensional dynamical system.

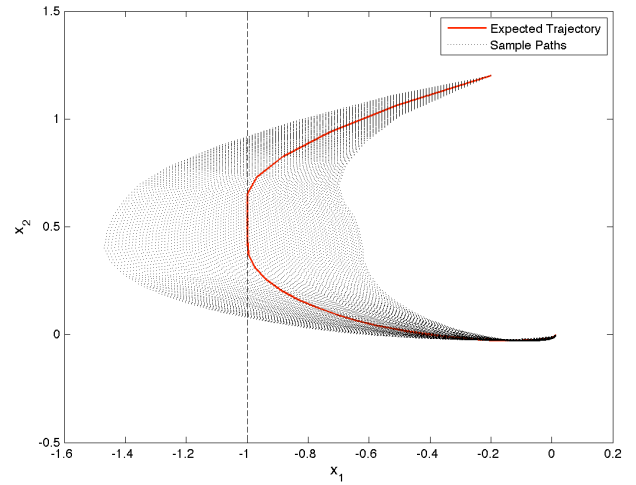


Fig. 1. State trajectories with expectation constraints.

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